MATH151, Fall 2022
Common Exam III - Version B

LAST NAME (print): ________________________ FIRST NAME (print): ________________________

INSTRUCTOR: ________________________ UIN: ________________________

SECTION NUMBER: ____________

DIRECTIONS:

- No calculators, cell phones, smart watches, headphones, or other electronic devices may be used, and must be put away.
- TURN OFF cell phones and put them away. If a cell phone is seen during the exam, your exam will be collected and you will receive a zero.
- In Part I, mark the correct choice on your ScanTron using a No. 2 pencil. The scantrons will not be returned, therefore *for your own records, also record your choices on your exam!*
- In Part II, present your solutions in the space provided. Show all your work neatly and concisely, and clearly indicate your final answer. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.
- **Be sure to fill in your name, UIN, section number, and version letter of the exam on the ScanTron form.**

THE AGGIE HONOR CODE

"An Aggie does not lie, cheat or steal, or tolerate those who do."

Signature: ____

Part I: Multiple Choice. 4 points each

1. Find $\lim_{x\to\infty} x\tan\left(\frac{5}{x}\right)$.

   (a) 5 ← correct

   (b) 0

   (c) 1

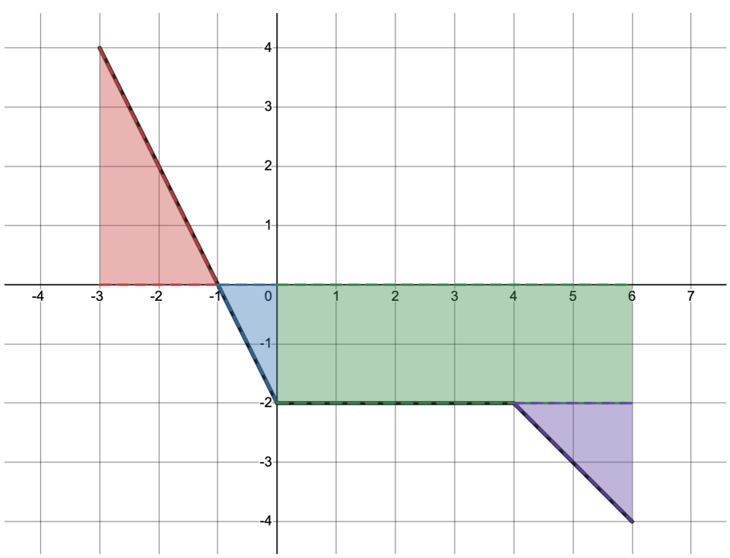
   (d) ∞

   (e) $\frac{1}{5}$

   Solution.

$$\lim_{x\to\infty} x\tan\left(\frac{5}{x}\right) = \lim_{x\to\infty}\frac{\tan(5/x)}{1/x} \overset{L}{=} \lim_{x\to\infty}\frac{\sec^2(5/x)(-5/x^2)}{-1/x^2} = \lim_{x\to\infty} 5\sec^2(5/x) = 5\sec^2(0) = 5$$

2. Use the graph of f below to evaluate $\int_{-3}^{6} f(x)\,dx$.



   (a) -7

   (b) -11 ← correct

   (c) -16

   (d) 19

   (e) -10

Solution. $\int_{-3}^{6} f(x)\,dx = 4 - 1 - 12 - 2 = -11$

3. Find the value(s) of c that satisfy the conclusion of the Mean Value Theorem for the $f(x) = x^3 - x$ on the interval $[0, 3]$.

   (a) $\sqrt{\frac{7}{3}}$

   (b) $\frac{3}{2}$

   (c) 1

   (d) $\sqrt{\frac{10}{3}}$

   (e) $\sqrt{3}$ ← correct

Solution. Note $f'(x) = 3x^2 - 1$.

$$\begin{aligned} f'(c) &= \frac{f(3) - f(0)}{3 - 0} \\ 3c^2 - 1 &= \frac{24 - 0}{3} \\ 3c^2 - 1 &= 8 \\ 3c^2 &= 9 \\ c^2 &= 3 \\ c &= \pm\sqrt{3} \implies c = \sqrt{3} \end{aligned}$$

4. Suppose $f(x)$ has a domain of all real numbers and $f''(x) = x^3(x-2)^5(x^2 + 6x - 16)$. Find the x-coordinates of the inflection points of f.

   (a) $x = -8$ only

   (b) $x = 0$ only

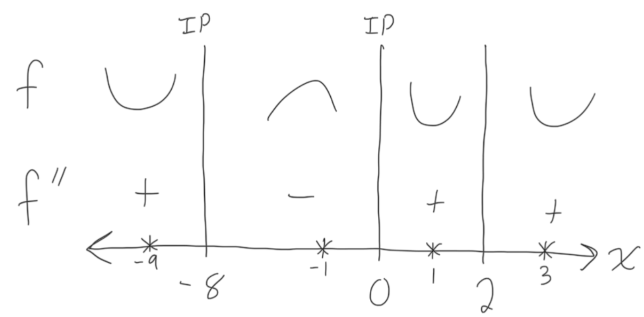
   (c) $x = -8$ and $x = 0$ ← correct

   (d) $x = -8$, $x = 0$, and $x = 2$

   (e) $x = 0$ and $x = 2$

Solution.

$$f''(x) = x^3(x-2)^5(x-2)(x+8) = x^3(x-2)^6(x+8) = 0$$
$$\implies x = 0, 2, -8$$



5. Find $\lim_{x\to 0} \frac{e^{3x} - 3x - \cos(x)}{2x^3 + x^2}$.

   (a) 5 ← correct

   (b) 0

   (c) ∞

   (d) 2

   (e) $\frac{7}{2}$

Solution.

$$\lim_{x\to 0} \frac{e^{3x} - 3x - \cos(x)}{2x^3 + x^2} \overset{L}{=} \lim_{x\to 0} \frac{3e^{3x} - 3 + \sin(x)}{6x^2 + 2x} \overset{L}{=} \lim_{x\to 0} \frac{9e^{3x} + \cos(x)}{12x + 2} = \frac{9+1}{2} = 5$$

6. Set up the limit to find the area under the graph of $f(x) = \sqrt{x+3}$ on $[1, 4]$.

(a) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\sqrt{3 + \frac{3i}{n}}$

(b) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\sqrt{1 + \frac{3i}{n}}$

(c) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n}\sqrt{4 + \frac{4i}{n}}$

(d) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\sqrt{4 + \frac{3i}{n}}$ ← correct

(e) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n}\sqrt{3 + \frac{4i}{n}}$

Solution. $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$ where $\Delta x = \frac{4-1}{n} = \frac{3}{n}$ and $x_i^* = 1 + i\left(\frac{3}{n}\right) = 1 + \frac{3i}{n}$. Thus

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\left(1 + \frac{3i}{n}\right) + 3} \cdot \frac{3}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\sqrt{4 + \frac{3i}{n}}$$

7. Find the absolute maximum and minimum values of $f(x) = 6\sqrt{x} - x + 1$ on $[0, 25]$.

(a) Absolute maximum is 10; absolute minimum is 1 ← correct

(b) Absolute maximum is 10; absolute minimum is 6

(c) Absolute maximum is 9; absolute minimum is 0

(d) Absolute maximum is 6; absolute minimum is 1

(e) Absolute maximum is 10; absolute minimum is 0

Solution.

$$f'(x) = \frac{3}{\sqrt{x}} - 1 = \frac{3 - \sqrt{x}}{\sqrt{x}} \implies x = 0, 9 \text{ are critical numbers.}$$

$$f(0) = 0 - 0 + 1 = 1 \text{ (min)}$$

$$f(9) = 18 - 9 + 1 = 10 \text{ (max)}$$

$$f(25) = 30 - 25 + 1 = 6$$

8. Find the critical numbers of $f(x) = x^{2/5}(x-6)^2$.

(a) $x = 1, 6$

(b) $x = 0, 6$

(c) $x = 6, 12$

(d) $x = 0, 1, 6$ ← correct

(e) $x = 0, 6, 12$

Solution.

$$
\begin{aligned}
f'(x) = \frac{2}{5}x^{-3/5}(x-6)^2 + x^{2/5}\big[2(x-6)\big]\cdot\frac{5x^{3/5}}{5x^{3/5}} &= \frac{2(x-6)^2}{5x^{3/5}} + \frac{10x(x-6)}{5x^{3/5}} \\
&= \frac{2(x-6)\big[(x-6)+5x\big]}{5x^{3/5}} \\
&= \frac{2(x-6)(6x-6)}{5x^{3/5}}
\end{aligned}
$$

$f'(x) = 0 \implies (x-6)(6x-6) = 0 \implies x = 6, 1$
$f'(x)$ DNE $\implies 5x^{3/5} = 0 \implies x = 0$

9. An object is traveling at 10 m/s when it starts to accelerate at 6 m/s^2. How far does the object travel before reaching a speed of 40 m/s?

(a) 125 m ← correct

(b) 50 m

(c) 150 m

(d) 110 m

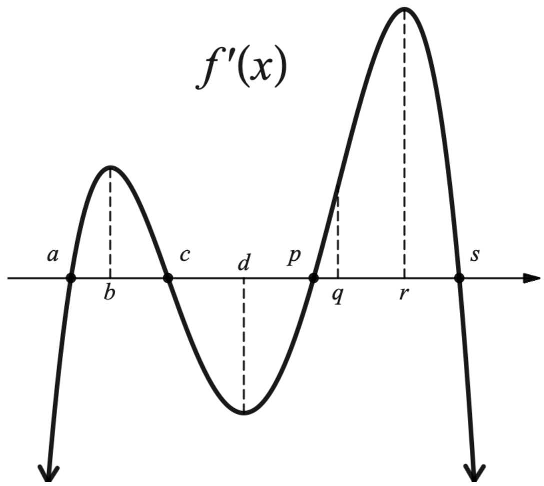
(e) 200 m

Solution. We have $a(t) = 6$ and $v(0) = 10$, and assume $s(0) = 0$.

$$
\begin{aligned}
& a(t) = 6 \\
\implies\ & v(t) = 6t + C \\
\implies\ & v(0) = 0 + C = 10 \implies C = 10 \\
\implies\ & s(t) = 3t^2 + 10t + D \\
\implies\ & s(0) = 0 + 0 + D = 0 \implies D = 0
\end{aligned}
$$

Thus $s(t) = 3t^2 + 10t$. Note $v(t) = 6t + 10 = 40$ when $t = 5$ and $s(5) = 125$.

The graph below is the **derivative**, $f'(x)$, of a continuous function f whose domain is all real numbers. Use this graph to answer questions 10 and 11.



10. Find the value(s) of x where f has a local maximum.

   (a) $x = a$ and $x = p$
   (b) $x = c$ and $x = s$ ← correct
   (c) $x = b$ and $x = r$
   (d) $x = d$
   (e) Cannot be determined.

   Solution. f has a local maximum when f' changes from positive to negative (i.e., f changes from increasing to decreasing). This is at $x = c$ and $x = s$.

11. Find the interval(s) where f is concave up.

   (a) (b, d), (r, ∞)
   (b) (a, c), (p, s)
   (c) $(-\infty, b)$, (d, r) ← correct
   (d) (c, q)
   (e) (c, p)

   Solution. f is concave up when f' is increasing. This is on $(-\infty, b)$ and (d, r).

12. Find the interval(s) where $g(x) = e^x(x^2 - x - 5)$ is decreasing.

(a) $(-\infty, -3)$, $(2, \infty)$

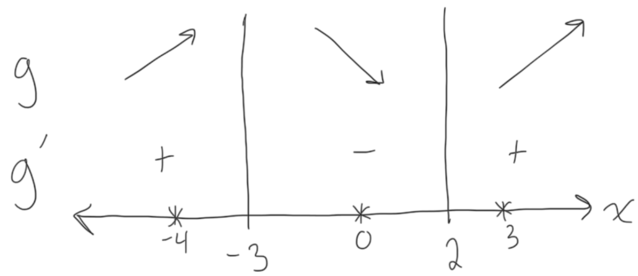
(b) $(-3, 0)$, $(0, 2)$

(c) $(-\infty, -3)$, $(0, \infty)$

(d) $(-3, 2)$ ← correct

(e) $(-3, \infty)$

Solution. $g'(x) = e^x(x^2 - x - 5) + e^x(2x - 1) = e^x(x^2 + x - 6) = e^x(x + 3)(x - 2) = 0 \implies x = -3, 2.$



13. Given $\int_{-1}^{2} f(x)\,dx = 1$, $\int_{2}^{5} f(x)\,dx = 4$ and $\int_{5}^{-1} g(x)\,dx = -5$, find $\int_{-1}^{5} [f(x) + 3g(x)]\,dx$.

(a) 20 ← correct

(b) −10

(c) 10

(d) 19

(e) −11

Solution.

$$\begin{aligned}\int_{-1}^{5} [f(x) + 3g(x)]\,dx = \int_{-1}^{5} f(x)\,dx + 3\int_{-1}^{5} g(x)\,dx &= \left[\int_{-1}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx\right] - 3\int_{5}^{-1} g(x)\,dx \\ &= 1 + 4 - 3(-5) \\ &= 20\end{aligned}$$

14. Which of the following is **true**?

(a) Every function attains an absolute minimum and absolute maximum on a closed interval.
(b) A left Riemann sum is always an underestimate.
(c) Any limit of the form $\infty - \infty$ evaluates to 0.
(d) $G(x) = \arccos x$ is the only antiderivative of $g(x) = \dfrac{-1}{\sqrt{1-x^2}}$.
(e) If $f'(-3) = 0$ and $f''(-3) = 4$, then f has a local minimum at $x = -3$. ← correct

Solution. (e) is true by the 2^{nd} Derivative Test: $f'(-3) = 0$ and $f''(-3) = 4 > 0$ means a horizontal tangent and concave up at $x = -3$, giving a local minimum.
(a) is false as we need a *continuous* function (e.g., $f(x) = \begin{cases} x+1 & \text{if } -1 \le x < 0 \\ x-1 & \text{if } 0 \le x \le 1 \end{cases}$ does not have an absolute maximum)
(b) is false as L_n can be an overestimate (when the function is decreasing) or neither
(c) is false as $\infty - \infty$ is indeterminate and can take on any value as well as $\pm\infty$ (e.g., $\lim\limits_{x\to\infty}(x^2 - x)$)
(d) is false as $\arccos x + C$ and $-\arcsin x + D$ are all antiderivatives of g

15. Suppose $f'(x) = \dfrac{2x + \sqrt[3]{x^4} - \sqrt{x}}{x}$. Find $f(0)$ if $f(1) = 2$.

(a) $\frac{13}{4}$
(b) $\frac{2}{3}$
(c) $\frac{8}{3}$
(d) $\frac{5}{4}$ ← correct
(e) 0

Solution.

$$\begin{aligned} f'(x) &= \frac{2x}{x} + \frac{x^{4/3}}{x} - \frac{x^{1/2}}{x} \\ &= 2 + x^{1/3} - x^{-1/2} \end{aligned}$$

$$\implies f(x) = 2x + \frac{3}{4}x^{4/3} - 2x^{1/2} + C$$

$$\implies f(1) = 2 + \frac{3}{4} - 2 + C = 2 \implies C = \frac{5}{4}$$

$$\implies f(x) = 2x + \frac{3}{4}x^{4/3} - 2x^{1/2} + \frac{5}{4} \implies f(0) = \frac{5}{4}$$

.

16. Estimate the area under the graph of $f(x) = 25 - x^2$ from $x = -2$ to $x = 4$ using 3 rectangles of equal width and midpoints.

(a) 132
(b) 134
(c) 128 ← correct
(d) 110
(e) 192

Solution. $\Delta x = \dfrac{4-(-2)}{3} = 2$ and

$$M_3 = \Delta x \sum_{i=1}^{3} f(x_i^*) = 2[f(-1)+f(1)+f(3)] = 2[24+24+16] = 128$$

Part II: Work Out Problems

Directions: Present your solutions in the space provided. *Show all your work* neatly and concisely and *box your final answer.* You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

17. (6 points) Find the most general antiderivative of $f'(x) = \sin x + \dfrac{3}{x} + \dfrac{8}{1+x^2} + 2^x + \sec x \tan x$.

Solution.

$$f(x) = -\cos x + 3\ln|x| + 8\arctan x + \frac{2^x}{\ln 2} + \sec x + C$$

18. (9 points) Find $\displaystyle\lim_{x\to 0^+} (\cos x)^{7/x^2}$.

Solution. $y = (\cos x)^{7/x^2} \implies \ln y = \dfrac{7}{x^2} \cdot \ln(\cos x)$

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{7\ln(\cos x)}{x^2} \overset{L}{=} \lim_{x\to 0^+} \frac{7\frac{-\sin x}{\cos x}}{2x} = \lim_{x\to 0^+} \frac{-7\tan x}{2x} \overset{L}{=} \lim_{x\to 0^+} \frac{-7\sec^2 x}{2} = -\frac{7}{2}$$

$$\lim_{x\to 0^+} (\cos x)^{7/x^2} = e^{-7/2}$$

19. (10 points) Consider the function $f(x) = \dfrac{x+3}{(x-2)^2}$, for which $f'(x) = \dfrac{-x-8}{(x-2)^3}$ and $f''(x) = \dfrac{2x+26}{(x-2)^4}$.

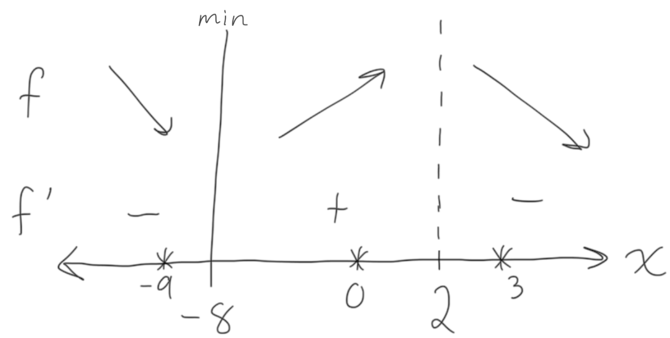
(a) What is the domain of f?

Solution. We need $x - 2 \neq 0$, so $x \neq 2$.

domain: $\boxed{(-\infty, 2) \cup (2, \infty)}$

(b) Determine the interval(s) on which f is increasing or decreasing. If there are none, write DNE.

Solution. $f'(x) = 0 \implies -x - 8 = 0 \implies x = -8$



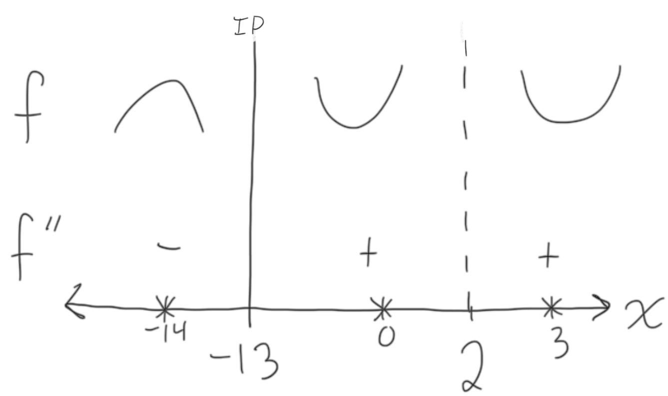
increasing: $\boxed{(-8, 2)}$ decreasing: $\boxed{(-\infty, -8),\ (2, \infty)}$

(c) Determine the x-coordinates of any local extrema. If there are none, write DNE.

local maximum at $x = \boxed{\text{DNE}}$ local minimum at $x = \boxed{-8}$

(d) Determine the interval(s) on which f is concave upward or concave downward. If there are none, write DNE.

Solution. $f''(x) = 0 \implies 2x + 26 = 0 \implies x = -13$



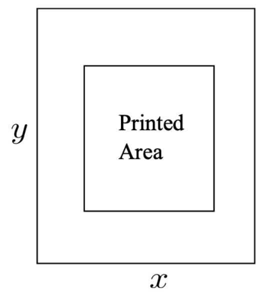
concave up: $\boxed{(-13, 2),\ (2, \infty)}$ concave down: $\boxed{(-\infty, -13)}$

(e) Determine the x-coordinates of any inflection points. If there are none, write DNE.

inflection point at $x = \boxed{-13}$

20. (11 points) The top and bottom margins of a poster are each 3 in and the side margins are 1 in. The poster is to have a total area of 108 in^2. Find the dimensions of the poster that will maximize the printed area. Justify that your answer gives a maximum.

Solution 1. Let x and y be the sides of the *poster* as pictured below.



Maximize: $P = (x-2)(y-6)$

Constraint: $xy = 108$

$$y = \frac{108}{x}$$

$$\Longrightarrow P(x) = (x-2)\left(\frac{108}{x} - 6\right) = 108 - 6x - \frac{216}{x} + 12$$

Calculus: $P'(x) = -6 + \dfrac{216}{x^2} = 0$

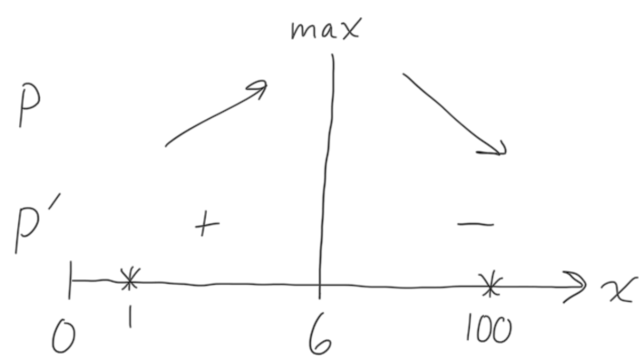
$$\frac{216}{x^2} = 6$$

$$216 = 6x^2$$

$$36 = x^2$$

$$x = 6$$

We can verify this is a maximum by using the 1^{st} Derivative Test:



Alternatively, we can use the 2^{nd} Derivative Test: $P''(x) = -\dfrac{432}{x^3} < 0$ for $x > 0$, so P is concave down on its domain and the local maximum at $x = 6$ must be the absolute maximum.

The dimensions of the poster that maximize the printed area are $\boxed{x = 6 \text{ in and } y = \dfrac{108}{6} = 18 \text{ in}}$.

Solution 2. Let x and y be the sides of the *printed area.*

$$\begin{aligned}
\text{Maximize: } & P = xy \\
\text{Constraint: } & (x+2)(y+6) = 108 \\
& y = \frac{108}{x+2} - 6 \\
\Longrightarrow & P(x) = x\left(\frac{108}{x+2} - 6\right) = \frac{108x}{x+2} - 6x \\
\text{Calculus: } & P'(x) = \frac{(x+2)108 - 108x}{(x+2)^2} - 6 = \frac{216}{(x+2)^2} - 6 = 0 \\
& \frac{216}{(x+2)^2} = 6 \\
& 36 = (x+2)^2 \\
& 6 = x+2 \\
& x = 4
\end{aligned}$$

We can verify this is a maximum by using the 1^{st} Derivative Test: $P'(x) > 0$ on $(0,4)$ and $P'(x) < 0$ on $(4,\infty)$. Alternatively, we can use the 2^{nd} Derivative Test: $P''(x) = -\dfrac{432}{(x+2)^3} < 0$ for $x > 0$, so P is concave down on its domain and the local maximum at $x = 4$ must be the absolute maximum.

The dimensions of the poster that maximize the printed area are $\boxed{x+2 = 6 \text{ in and } y+6 = \frac{108}{6} = 18 \text{ in}}$.